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DIFFERENTIAL GAME OF GUIDANCE FOR A SYSTEM WITH SLACK AND INTEGRAL CONSTRAINTS

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The problem of guidance onto a convex target set of a system with slack is analyzed on the assumption that the realization of controls of the first player is hindered by integral constraints. Sufficient conditions of the problem solvability are formulated and an example is presented. This paper is related to [1 — 4].

1. Let us consider a controlled system described by the following vector differential equation:

$$\frac{dx}{dt} = A(t)x + C(v)u, \quad C(v) = \begin{Vmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ \cos v & -\sin v \\ \sin v & \cos v \end{Vmatrix} \tag{1.1}$$

Here x is the n-dimensional phase vector of the system, u is two-dimensional control vector of the first player and v is the control of the second player. The realizations of the player controls are restricted by the conditions

$$\int_t^{\vartheta} \| u[\tau] \|^2 d\tau \leqslant \mu^2[t], \quad v[t] \in [-\alpha, +\alpha]$$

for any $t \in [t_0, \vartheta]$, $(\alpha < \pi/2)$.

The symbol $\| \cdot \|$ denotes the norm in the corresponding Euclidean space, $\mu[t]$ are the constraints imposed on the resources of the control of the first player, and ϑ denotes a fixed instant of time.

The variation of the constraint $\mu[t]$ is determined by the amount of resource used up in the course of the game, i.e.

$$\mu^2[t+\Delta] = \mu^2[t] - \int_t^{t+\Delta} \|u[\tau]\|^2 d\tau \quad (\Delta > 0) \tag{1.2}$$

Equation (1.2) has an associated differential equation

$$d\mu^2/dt = -\|u[t]\|^2 \tag{1.3}$$

We shall assume that the inequality

$$\int_t^{t+\Delta} \left[\left(\sum_{j=1}^n l_j x_{jn-1}[\vartheta;\tau]\right)^2 + \left(\sum_{j=1}^n l_j x_{jn}[\vartheta;\tau]\right)^2\right] d\tau > 0 \tag{1.4}$$

holds for any t, Δ and l ($t \in [t_0, \vartheta]$, $0 \leqslant \Delta < \vartheta - t$, $\|l\| = 1$), where $x_{jk}[\vartheta;\tau]$ is an element of the fundamental matrix $X[\vartheta;\tau]$ of solutions of the equation $dx/d\tau = A(\tau)x$. We note that the inequality (1.4) implies that the system (1.1) is fully controllable in u.

We define a bounded closed and convex target set M in the space of vectors x. The game takes place within a given interval of time $[t_0, \vartheta]$. The payoff of the game is determined by the equation $\gamma[x[\vartheta]] = \rho[x[\vartheta], M]$, where $\rho[x[\vartheta], M]$ denotes the distance between the final state $x[\vartheta]$ of the system (1.1) and the set M in the Euclidean metric. The vector $\{t, z\} = \{t, x, \mu^2\}$ is understood to define the position of the game, and we shall consider the motion $z[t]$ in the space R^{n+1} of vectors $z = \{x, \mu^2\}$.

Let the initial position $\{t_0, x_0, \mu_0^2\}$ of the game be fixed. The aim of the first player is to attain the smallest possible payoff of the game which starts from the position $\{t_0, x_0, \mu_0^2\}$. The second player tries to prevent the first player from achieving his aim. We assume that at any instant of time t each player knows the exact value of the position $\{t, x, \mu^2\}$ of the game; neither player knows the future positions, nor the location of the opponent.

Let us now compare the system of equations (1.1), (1.2) and initial conditions $x[t_0]$ and $\mu^2[t_0]$, with the system of differential equations (1.1), (1.3) written in the form of a vector differential equation as follows:

$$dz/dt = f(t, z, u, v) \tag{1.5}$$

Here the symbol $f(t, z, u, v)$ denotes the right-hand side of the system (1.1), (1.3).

2. We shall solve the problem from the viewpoint of the first player, using the following additional definitions.

Definition 1. We shall call the function $U = U(t, z)$, the admissible strategy of the first player. This strategy places each position $\{t, z\}$ in correspondence with some closed convex set $U(t, z)$ ($U(t, z) \subset R^2$), which is upper semicontinuous relative to inclusion over the set t, z. The strategy is also such, that for any closed region $D \subseteq Q$ and for every point $\{t, z\}$ belonging to D there exists a summable function $B(\tau)$ ($\tau \in [t, \vartheta)$) which satisfies at the points $\{\tau, z'\}$ of D the condition: if $u \in U(\tau, z')$, then $\|u\|^2 \leqslant B(\tau)$. Here

$Q = [t_0, \vartheta) \times Z$, $Z = \{z = \{x, \mu^2\} : \|x - x_0\| \leqslant L,\ 0 \leqslant \mu^2 \leqslant \mu_0^2\}$, and L is a fixed sufficiently large positive number.

Let us set $U(t, z) = \{0\}$ for $\mu \leqslant 0$.

Definition 2. We define, as a motion generated by an admissible strategy $U = U(t, z)$ on the interval $[t_0, t_*]$ and emerging from the position $\{t_0, z_0\}$, any absolutely continuous vector function $z[t] = z[t; t_0, z_0; U]$ which satisfies the initial condition $z[t_0] = z_0$ as the equation

$$dz[t]/dt = f(t, z[t], u[t], v[t])$$
$$u[t] \in U(t, z[t]), \quad v[t] \in [-\alpha, +\alpha]$$

for almost every $t \in [t_0, t_*]$ $(t_0 < t_* < \vartheta)$, where $v[t]$ is a function measurable on $[t_0, \vartheta]$ satisfying the inclusion defined above.

The existence of these motions follows from the results of [5].

The motion $z[t] = z[t; t_0, z_0; U]$ is defined on any interval $[t_0, t_*]$ $(t_* < \vartheta)$, therefore it can be extended continuously to the instant $t = \vartheta$. This defines the motions $z[t] = z[t; t_0, z_0; U]$ on the interval $[t_0, \vartheta]$.

Let us introduce the functional

$$\rho_U{}^*[t_0, z_0; \vartheta] = \max_{x[\vartheta]=x[\vartheta;\, t_0,\, z_0;\, U]} \rho[x[\vartheta]; M]$$
$$(\{x[\vartheta], \mu^2[\vartheta]\} = z[\vartheta])$$

We formalize the problem of guidance stated in Sect. 1, in the following manner.

Problem. From amongst the admissible strategies $U(t, z)$ we require to find the optimal minimax strategy $U^0(t, z)$ satisfying the relation

$$\min_U \rho_U{}^*[t_0, z_0; \vartheta] = \rho_{U^\circ}{}^*[t_0, z_0; \vartheta]$$

We shall construct the strategy $U^\circ(t, z)$ using an auxiliary program. Below we define the elements of this program.

We define, as the admissible countercontrol $V(t, u)$ of the second player, a single-valued Borel function on the set (t, u) which, for each pair $(t, u) \in [t_0, \vartheta] \times R^2$ has a corresponding value $V(t, u) \in [-\alpha, +\alpha]$. Let

$$\{t_*, z_*\} = \{t_*, x_*, \mu_*{}^2\} = \{t_*, x[t_*], \mu^2[t_*]\} (t_* \in [t_0, \vartheta])$$

be some position of the game. We define, as a motion generated by an admissible countercontrol $V(t, u)$ of the second player, the vector function $x[t] = x[t; t_*, z_*; V]$ $(t \in [t_*, \vartheta])$ which is absolutely continuous on $[t^*, \vartheta]$ and satisfies the relation

$$x[t] = \sum_{i=1}^{n+1} \lambda_i[t] x^{(i)}[t]$$

$$\sum_{i=1}^{n+1} \lambda_i[t] = 1, \quad \lambda_i[t] \geqslant 0 \quad \text{for } t \in [t_*, \vartheta]$$

$$x^{(i)}[t] = \lim_{k \to \infty} x_k^{(i)}[t]$$

$$x_k^{(i)}[t] = X[t; t_*] x[t_*] + \int_{t_*}^{t} X[t; \tau] C(V(\tau, u_k^{(i)}[\tau])) u_k^{(i)}[\tau] d\tau$$

Here the vector function $u_k^{(i)}[\tau]$ considered on $[t_*, \vartheta]$ belongs to the set $\Phi[t_*, \mu[t_*]]$ of all functions $u[\tau]$ ($\tau \in [t_*, \vartheta]$) satisfying the inequality

$$\int_{t_*}^{\vartheta} \| u[\tau] \|^2 d\tau \leqslant \mu^2 [t_*], \qquad z_* = \{x[t_*], \mu^2[t_*]\}$$

We shall call the set $\{x[\vartheta]: x[\vartheta] = x[\vartheta; t_*; z_*, V]\}$ the region of attainability $G(t_*, z_*; \vartheta, V)$ in the space x for the motions $x[t] = x[t; t_*, z_*; V]$ from the position $\{t_*, z_*\}$ at the instant ϑ.

We denote by M_ε the closed Euclidean ε-neighborhood of the set M, and by $\varepsilon_0 = \varepsilon_0(t_*, z_*; \vartheta)$ the lower bound of the values $\varepsilon \geqslant 0$ for which at least one motion $x[t] = x[t; t_*, z_*; V]$ attains M_ε, at the instant ϑ irrespective of the choice of the admissible countercontrol $V(t, u)$. Let $G = G(t_*, z_*; \vartheta; V)$ be the region of attainability, the distance of which from the set M is $\varepsilon_0(t_*, z_*; \vartheta)$. The region of attainability G intersects the set M_ε if and only if the closed $(-M_\varepsilon)$ neighborhood of the region G contains the point $x = 0$. The region $G_{(-M_\varepsilon)}$ is composed of all vectors $q = g + k - h$ where $g \in G(t_*, z_*; \vartheta; V)$, $h \in M$ and $\| k \| \leqslant \varepsilon$. The bounded closed convex set $G_{(-M_\varepsilon)}$ represents the intersection of all its reference half-spaces (g^* is the limit element of G)

$$l' g^* \geqslant \min_q l' q \tag{2.1}$$

$$q = x[\vartheta] + k - h = X[\vartheta; t_*] x_* + \int_{t_*}^{\vartheta} X[\vartheta; \tau] w[\tau] d\tau + k - h$$

where

$$w[\tau] = \sum_{j=1}^{n+1} \lambda_j[\tau] w^{(j)}[\tau], \qquad \tau \in [t_*, \vartheta] \tag{2.2}$$

$$\sum_{j=1}^{n+1} \lambda_j[\tau] = 1, \quad \lambda_j[\tau] \geqslant 0, \quad j = 1, 2, \ldots, n+1 \tag{2.3}$$

$w^{(j)}[\tau]$ is a weak limit on $[t_*, \vartheta]$ of a certain sequence of functions

$$\{C(V(\tau, u_k^{(j)}[\tau]))\, u_k^{(j)}[\tau]\}$$

where $u_k^{(j)}[\tau] \in \Phi[t_*, \mu[t_*]]$.

From the previous arguments it follows that

$$\varepsilon_0(t_*, z_*; \vartheta) = \sup_{\|l\|=1} \sup_V \min_{q \in G(t_*, z_*; \vartheta; V)} l' q =$$
$$\sup_{\|l\|=1} (\sup_V \min_{w[\cdot] \in F(V)} I(t_*) - l' X[\vartheta; t_*] x_* - \max_{h \in M} l' h)$$
$$I(t_*) = \int_{t_*}^{\vartheta} l' X[\vartheta; t] w[t] dt$$

where $F(V)$ is a set of functions $\{w[\cdot]\}$ satisfying the relations (2.2) and (2.3) on the interval $[t_*, \vartheta]$; $w[\cdot]$ denotes the function $w[\tau]$ considered on the interval $[t_*, \vartheta]$.

We shall show that the upper limit

$$\sup_V \min_{w[\cdot] \in F(V)} I(t_*) \tag{2.4}$$

can be attained for each fixed value of the vector l, on some admissible function $V_l(t, u)$. We first define the set

$$V_l^*(t, u) = \{v_u:\ v_u \in [-\alpha,\ +\alpha],\ l'X[\vartheta;\ t]C(v_u)u = \max_{v\in[-\alpha,+\alpha]} l'X[\vartheta;\ t]C(v)u\}$$

The set $V_l^*(t, u)$ is bounded closed and upper semicontinuous relative to inclusion with respect to the variables t, u, for every value of the vector l. This implies that there exists a single-valued Borel function $V_l(t, u)$ satisfying inclusion $V_l(t, u) \in V_l^*(t, u)$. This function belongs to the class of admissible countercontrols $V(t, u)$.

We note that the function $V_l(t, u)$ can be defined by the relations

$$V_l(t, u) = \begin{cases} \alpha, & \psi_l(t, u) \in [-\pi, -\alpha] \\ -\alpha, & \psi_l(t, u) \in [+\alpha, +\pi] \\ -\psi_l(t, u), & \psi_l(t, u) \in [-\alpha, +\alpha] \end{cases}$$

Here $\psi_l(t, u)$ denotes the angle between the vectors $\{l'X[\vartheta;\ t]\}^*$ and u; $\{l'X[\vartheta;\ t]\}^*$ is the projection of the vector $l'X[\vartheta;\ t]$ on the $(x_{n-1},\ x_n)$ coordinate space, and the angles $\psi_l(t, u)$ are counted from the vector $\{l'X[\vartheta;\ t]\}^*$.

The definition of the admissible countercontrol $V_l(t, u)$ implies that the following relations hold:

$$\inf_{u[\cdot]} \int_{t_*}^{\vartheta} l'X[\vartheta;\ t]\, C(V_l(t, u[t]))\, u[t]\, dt \geqslant$$

$$\inf_{u[\cdot]} \int_{t_*}^{\vartheta} l'X[\vartheta;\ t]\, C(V(t, u[t]))\, u[t]\, dt$$

$$u[\cdot] \in \Phi[t_*,\ \mu[t_*]]$$

or

$$\min_{w[\cdot]\in F(V_l)} I(t_*) \geqslant \min_{w[\cdot]\in F(V)} I(t_*)$$

Let $w_l[t]$ be a function belonging to $F(V_l)$ and satisfying the equation

$$I_l(t_*) = \min_{w[\cdot]\in F(V_l)} I(t_*),\quad I_l(t_*) = \int_{t_*}^{\vartheta} l'X[\vartheta;\ t]\, w_l[t]\, dt$$

From the last two relations it follows that the inequality is valid irrespective of the character of the admissible countercontrol $V_l(t, u)$, therefore

$$I_l(t_*) \geqslant \min_{w[\cdot]\in F(V)} I(t_*)$$

i.e. the upper bound (2.4) is attained on the admissible countercontrol $V_l(t, u)$. Taking into account the equation

$$J_l(t_*) = -\cos\alpha\ \mu[t_*]R(t_*)$$

$$R(t_*) = \left(\int_{t_*}^{\vartheta} \left[\sum\nolimits_{n-1n}^{2}(t) + \sum\nolimits_{nn}^{2}(t)\right] dt\right)^{1/2}$$

$$\Sigma_{kn}(t) = \sum_{j=1}^{n} l_j x_{jk}[\vartheta;\ t],\qquad k = n-1,\ n$$

we find that $\varepsilon_0 = \varepsilon_0(t_*,\ z_*;\ \vartheta)$ is given by the equation

$$\varepsilon_0 = \max_{\|l\|=1}\left(-\cos\alpha\mu[t_*]\, R(t_*) + l'X[\vartheta;\ t_*]\, x_* - \max_{h\in M} l'h\right) \tag{2.5}$$

when the right-hand side of (2.5) is nonnegative, otherwise $\varepsilon_0(t_*, z_*; \vartheta) = 0$. When $\varepsilon_0 > 0$, the maximum in the right-hand side of (2.5) is attained on a unique vector l°, since the minimizing quantity taken with the opposite sign is a function convex with respect to the variable l. We have the regular case. As the result, we find that a region $G(t_*, z_*; \vartheta; V_{l^\circ})$ exists which touches the ε_0-neighborhood of the set M. Let $g_0 = x[\vartheta; t_*, z_*; V_{l^\circ}]$ be the point of contact between the region $G(t_*, z_*; \vartheta; V_{l^\circ})$ and M_{ε_0}. It follows that the motion $x[t; t_*, z_*; V_{l^\circ}]$ which arrives at this point at the instant ϑ will be optimal, and the control $w^0[t] \in F(V_{l^\circ})$ generating this motion will satisfy the condition of the maximum principle

$$\int_{t_*}^{\vartheta} l^{\circ\prime} X[\vartheta; t] w^0[t]\, dt = \min_{w[\cdot]\in F(V)} \int_{t_*}^{\vartheta} l^{\circ\prime} X[\vartheta; t] w[t]\, dt \qquad (2.6)$$

Let us define the strategy $U_e(t_*, z_*)$ $(t_* < \vartheta)$ as follows:

$$U_e(t_*, z_*) = u_{l^\circ}[t_*] = \left\| \begin{matrix} u_{1l^\circ}[t_*] \\ u_{2l^\circ}[t_*] \end{matrix} \right\| \qquad (2.7)$$

$$u_{1l^\circ}[t_*] = \frac{-\mu[t_*]\, \Sigma_{n-1n}^{\circ}(t_*)}{R^\circ(t_*)}, \quad u_{2l^\circ}[t_*] = \frac{-\mu[t_*]\, \Sigma_{nn}^{0}(t_*)}{R^\circ(t_*)}$$

$$R^\circ(t_*) = \Big(\int_{t_*}^{\vartheta} [\Sigma_{n-1n}^{\circ 2}(t) + \Sigma_{nn}^{\circ 2}(t)]\, dt\Big)^{1/2}$$

$$\Sigma_{kn}^{\circ}(t) = \sum_{j=1}^{n} l_j^\circ x_{jk}[\vartheta; t], \qquad k = n-1, n$$

for the case $\varepsilon_0(t_*, z_*; \vartheta) > 0$. Here $l^\circ = l^\circ(t_*, z_*)$ is given by the expression

$$-\cos\alpha\, \mu[t_*] R^\circ(t_*) + l^{\circ\prime} X[\vartheta; t_*] x_* - \max_{h\in M} l^{\circ\prime} h = \varepsilon_0(t_*, z_*; \vartheta)$$

$U_e(t_*, z_*) = \{0\}$ in the case when $\varkappa < \mu_* \leqslant 0$, $(\varkappa < 0)$ or $\varepsilon_0(t_*, z_*; \vartheta) < 0$, and

$$U_e(t_*, z_*) = \mathrm{co}\,\{0 \cup u^*: \quad u^* = \lim_{k\to\infty} U_e(t_k, z_k), \qquad \lim_{k\to\infty}(t_k, z_k) = (t_*, z_*),\ \varepsilon_0(t_k, z_k; \vartheta) > 0 \quad \text{for} \quad k = 1, 2, \ldots\}$$

in the case when $\mu_* > 0$ and $\varepsilon_0(t_*, z_*; \vartheta) = 0$.

We can see that $U_e(t_*, z_*)$ is admissible by virtue of the inequality (1.4), and of the semicontinuity.

When the value of ϑ is fixed, the quantity ε_0 in the region $\varepsilon_0(t, z; \vartheta) > 0$ is a differentiable function of the variables t and z, and its partial derivatives are given by the equalities

$$\frac{\partial\varepsilon_0}{\partial t} = \frac{\mu[t]\cos\alpha}{R^\circ(t)} (\Sigma_{n-1n}^{\circ 2}(t) + \Sigma_{nn}^{\circ 2}(t)) - s[t]' A(t) x[t]$$

$$s[t]' = l^{\circ\prime} X[\vartheta; t], \quad \frac{\partial\varepsilon_0}{\partial x_i} = s_i[t], \quad ds[t]/dt = -A'(t) s[t]$$

The derivatives of ε_0 can be calculated according to the scheme (see [1]). The following relation also holds:

$$\max_{z[t]} \left(\frac{d\varepsilon_0(t, z[t]; \vartheta)}{dt}\right)_{U_e} = \min_U \max_{z[t]} \left(\frac{d\varepsilon_0(t, z[t]; \vartheta)}{dt}\right)_U = 0 \qquad (2.8)$$

Theorem. The extremal strategy $U_e(t, z)$ (2.7) of the first player guarantees to the first player the value of the payoff $\rho[x[\vartheta]; M]$ at the instant ϑ, equal to $\varepsilon_0(t_0, z[t_0]; \vartheta) = \min_U \rho^*_U(t_0, z[t_0]; \vartheta)$, provided that the initial position of the game t_0, $z[t_0]$ from which the game begins is such that $\varepsilon_0(t_0, z[t_0]; \vartheta) > 0$, and equal to zero when $\varepsilon_0(t_0, z[t_0]; \vartheta) \leqslant 0$.

Proof. The first case of $\varepsilon_0(t_0, z[t_0]; \vartheta) > 0$ is trivial. Let us consider the second case when $\varepsilon_0(t_0, z[t_0]; \vartheta) \leqslant 0$. Assume the opposite; let there exist $z[t]_{U_e}$ such that $\rho[x[\vartheta]_{U_e}, M] > 0$ $(z[\vartheta]_{U_e} = \{x[\vartheta]_{U_e} \mu^2[\vartheta]\}_{U_e}\})$.

We introduce the function $\varepsilon(t) = \varepsilon_0(t, z[t]_U; \vartheta)$, $\varepsilon(\vartheta) > 0$. Then there exists $t_* < \vartheta$ and a small Δ $(0 < \Delta < \vartheta - t_*)$ for which $\varepsilon(t_*) = 0$, $\varepsilon(t) > 0$ for all $t \in (t_*, t_* + \Delta]$. The following formula holds:

$$\varepsilon(t_* + \Delta) = \varepsilon(t_k) + \int_{t_k}^{t_*+\Delta} \frac{d\varepsilon(\tau)}{d\tau} d\tau, \quad t_k \in (t_*, t_* + \Delta] \tag{2.9}$$

Here $\{t_k\}$ converges to t_*. Since the function $\varepsilon(t)$ is continuous on $[t_*, t_* + \Delta]$, we have

$$\varepsilon(t_k) \to \varepsilon(t_*) = 0, \quad k \to \infty \tag{2.10}$$

The inequality $d\varepsilon(\tau)/d\tau \leqslant 0$ holds on every interval $[t_k, t_* + \Delta]$ by virtue of the relations (2.8), consequently we obtain

$$\int_{t_k}^{t_*+\Delta} \frac{d\varepsilon(\tau)}{d\tau} d\tau \leqslant 0 \tag{2.11}$$

The relations (2.9) — (2.11) yield the inequality $\varepsilon(t_* + \Delta) \leqslant 0$ which contradicts the condition of the theorem, and this completes the proof.

The strategy $U_e(t, z)$ realizes for the first player a result which coincides with the value $\varepsilon_0(t_0, z[t_0]; \vartheta) > 0$ and is optimal for the program constructed, therefore this strategy is the optimal minimax strategy.

3. Let us consider an example. A material point A of mass m moves in a three-dimensional Euclidean space R^3 along a circular orbit Γ_0 around the Earth. A material point B of mass m is guided into a sufficiently small neighborhood of the point A, at the instant t_0. We assume that the motion of the point B in the space R^3 of vectors $y = \{y_1, y_2, y_3\}$ is described by the following vector differential equation:

$$my^{\cdot\cdot} = f_g(y) + C(v)u \tag{3.1}$$

Here $f_g(y)$ is the force of attraction exerted on the point B in position $y = \{y_1, y_2, y_3\}$ by the Earth the center O of which coincides with the origin of the y_1, y_2, y_3 coordinate system, $v \in [-\alpha, +\alpha]$ is the interference which we shall regard as the control of the second player, and u denotes the two-dimensional control vector of the first player. We note that the control of the first player is subject to the integral constraint, described in Sect. 1 and equal to $\mu[t_0]$.

Our aim is to choose a control u which would ensure that the points A and B coincide within the period T of revolution of the point A about the Earth. Taking into account the fact that the orbit Γ_0 lies in the (y_1, y_3)-plane, we introduce the following generalized coordinates of the point B which characterize the deviation of the point B from A: $x_1 = \|y_B\| - \|y_A\|$; x_3 is the angle between the projections $y_A{}^*$ and $y_B{}^*$ of the vectors y_A and y_B on the (y_1, y_3)-plane, and $x_5 = y_{2B} - y_{2A}$. Here $y_A = \{y_{1A},$

$y_{2A}, y_{3A}\}$ and $y_B = \{y_{1B}, y_{2B}, y_{3B}\}$ are the vectors defining the positions of the points A and B in the space R^3.

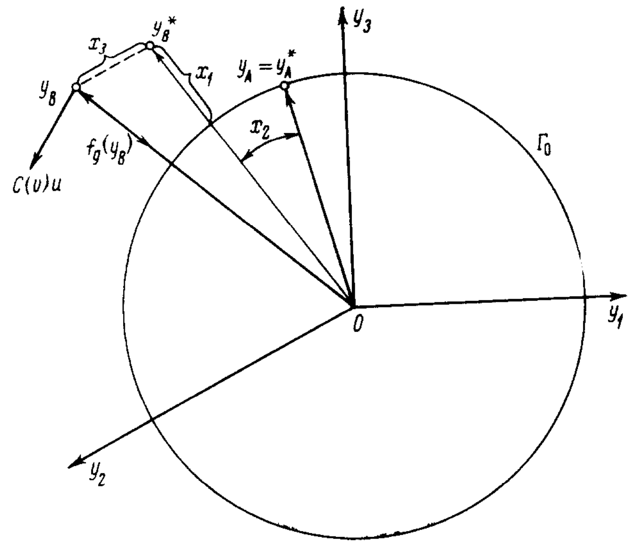

Fig. 1

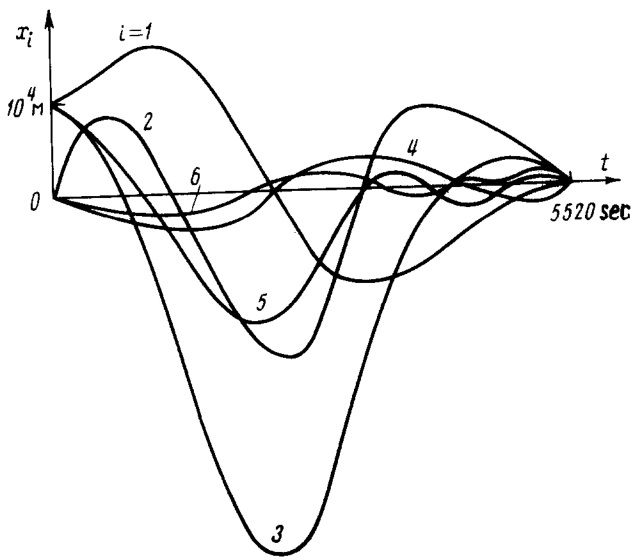

Fig. 2

We take the six-dimensional vector $x = \{x_1, x_2 = x_1^{\cdot}, x_3, x_4 = x_3^{\cdot}, x_5, x_6 = x_5^{\cdot}\}$ as the phase vector of the point B. Then Eq. (3.1) of motion of B reduces, in the linear approximation, to the following system of differential equations:

$$x_1^{\cdot} = x_2, \quad x_2^{\cdot} = 3\beta^2 x_1 + 2\beta R_0 x_4, \quad x_3^{\cdot} = x_4 \tag{3.2}$$

$$x_4^{\cdot} = -\frac{2\beta}{R_0} x_2 + \frac{1}{mR_0}(u_1 \cos v - u_2 \sin v)$$

$$x_5^{\cdot} = x_6, \quad x_6^{\cdot} = -\beta^2 x_5 + \frac{1}{m}(u_1 \sin v + u_2 \cos v)$$

$$\left(\beta = \sqrt{\frac{\nu_0 M_e}{R_0^3}}, \quad R_0 = \frac{Tv^*}{2\pi}\right)$$

where M_e is the mass of the Earth, ν_0 is the gravitational constant, and v^* is the orbital velocity. In solving the problem, we assumed that $T = \vartheta - t_0 = 5520$ sec, and $m = 200$ kg.

The problem was solved as follows: the quantity $\mu[t_0]$ was chosen so that $\varepsilon_0(t_0, z[t_0]; \vartheta) = 0$. The interval $[t_0, \vartheta]$ was divided into equal segments $[t_i, t_{i+1}]$ and the control of the first player was computed for each instant t_i using the formulas (2.3) where the vector l^0 was given by (2.4).

For the set of initial data

$$t_0 = 0 \text{ sec}, x_1[t_0] = x_3[t_0] = x_5[t_0] = 10^4 \text{ m}, \quad x_2[t_0] = x_6[t_0] = 0 \text{ m/sec},$$
$$x_4[t_0] = -1 \text{ m/sec}, v = 45°$$

the results obtained were as follows:

$$x_1[\vartheta] = -13\cdot 10^{-2} \text{ m}, \quad x_2[\vartheta] = -45\cdot 10^{-4} \text{ m/sec}, \quad x_3[\vartheta] = -5.7 \text{ m}$$
$$x_4[\vartheta] = -29\cdot 10^{-2} \text{ m/sec}, x_5[\vartheta] = 1.2 \text{ m}, \quad x_6[\vartheta] = -87\cdot 10^{-2} \text{ m/sec}$$

We note that the quantity x_3 was converted from radians to meters.

Figure 2 depicts the realization of the components x_i $(i = 1, \ldots, 6)$ of the phase vector of the system (3.2) for the initial conditions given above.

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